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New class of running-wave solutions of the $(2+1)$-dimensional sine–Gordon equation

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Abstract. A new class of running-wave solutions of the $(2+1)$-dimensional sine–Gordon equation is investigated. The obtained waves require two spatial dimensions for their propagation, i.e. they generalize solutions of the $(2+0)$-dimensional sine–Gordon equation. The parameters of the waves strongly depend on the wave amplitude and there exist forbidden areas for the wavenumber and frequency. The obtained solutions describe a new class of Josephson waves whose velocity is smaller than the Swihart velocity. If $\omega = 0$ the running waves are reduced to the self-consistent phase, current and magnetic field distributions in a large two-dimensional Josephson junction. The self-restriction coefficient for the Josephson current corresponding to one of the structures is calculated.

The (1+1)-dimensional sine–Gordon equation is well known because of its soliton solutions [1]. By means of this equation a lot of nonlinear phenomena—the waves in the long Josephson junction [2], the magnetic domain wall dynamics [3], the self-induced transparency [4] and so on [5–7]—can be described. On the basis of Lamb's ansatz [8] an approach for obtaining exact analytical solutions of the $(2+1)$-dimensional sine–Gordon equation

$$\frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} - \frac{\partial^2\varphi}{\partial t^2} = \sin\varphi \tag{1}$$

was developed [9]. Due to this approach a new class of running-wave solutions of (1) is investigated here. The obtained waves are expressed by the Jacobi elliptic function dn [10]:

$$\varphi_1 = 4\tan^{-1}\{A\,\mathrm{dn}[\alpha(y-y_0;k_1)]\,\mathrm{dn}[\beta(z-z_0)+\delta\gamma(t-t_0);k_2]\} \tag{2a}$$

$$k_1^2 = 1 - \frac{A^2-\alpha^2(1+A^2)}{\alpha^2A^2(1+A^2)} \tag{2b}$$

$$k_2^2 = 1 - \frac{A^2-(\beta^2-\gamma^2)(1+A^2)}{(\beta^2-\gamma^2)A^2(1+A^2)} \tag{2c}$$

$$\alpha^2 + (\beta^2-\gamma^2) = \frac{A^2}{1+A^2} \tag{2d}$$

$$\varphi_2 = 4\tan^{-1}\left\{A\frac{\mathrm{dn}[\alpha(y-y_0);k_1]}{\mathrm{dn}[\beta(z-z_0)+\delta\gamma(t-t_0);k_2]}\right\} \tag{3a}$$

$$k_1^2 = \frac{\alpha^2(1+A^2)^2 - A^2}{\alpha^2 A^2(1+A^2)} \tag{3b}$$

$$k_2^2 = 1 - \frac{A^2[1-(\beta^2-\gamma^2)(1+A^2)]}{(\beta^2-\gamma^2)(1+A^2)} \tag{3c}$$

$$\alpha^2 = (\beta^2-\gamma^2)A^2 \tag{3d}$$

where y_0, z_0, t_0 are constants, A is the amplitude of the wave, β is the dimensionless wavenumber, γ is the dimensionless frequency and α is a parameter connected with the period of the waves in the direction perpendicular to the direction of spreading. $\delta = \pm 1$ and $k_{1,2}$ are the modules of the Jacobi elliptic function [10].

Including the modules $k_{1,2}$ of the Jacobi elliptic functions we have six parameters among which three relations exist. There are also an additional three relations:

$$\gamma^2 - \beta^2 < 0 \tag{4}$$

$$0 \leqslant k_1 \leqslant 1 \tag{5}$$

$$0 \leqslant k_2 \leqslant 1. \tag{6}$$

The six relations impose restrictions on the parameters of the waves. We shall investigate the dependence of the wavenumber β, the frequency γ and the parameter α on the amplitude A of the wave. Equations (4)–(6) lead to allowed and forbidden areas for the wave parameters. For the wave φ_1 the restrictions are

$$\frac{A^2}{(1+A^2)^2} \leqslant \alpha^2 \leqslant \frac{A^2}{(1+A^2)} \tag{7a}$$

$$\frac{A^2}{(1+A^2)^2} \leqslant \beta^2 - \gamma^2 \leqslant \frac{A^2}{1+A^2}. \tag{7b}$$

By means of (2*d*) we can express γ by β and A, and then (7*b*) leads to an additional restriction for the parameter α:

$$0 \leqslant \alpha^2 \leqslant \frac{A^4}{(1+A^2)^2}. \tag{8}$$

Finally, the allowed area for α is

$$\frac{A^2}{(1+A^2)^2} \leqslant \alpha^2 \leqslant \frac{A^4}{(1+A^2)^2}. \tag{9}$$

Equation (9) shows that the amplitude of the wave (2) must be $A \geqslant 1$.

For the wave φ_2 the allowed areas for the parameter α are

$$\frac{A^2}{(1+A^2)^2} \leqslant \alpha^2 \leqslant \frac{A^2}{1+A^2} \tag{10}$$

$$\frac{A^2}{(1+A^2)^2} \leqslant \beta^2 - \gamma^2 \leqslant \frac{1}{1+A^2}. \tag{11}$$

From (3d) we have that $\beta^2 - \gamma^2 = \alpha^2/A^2$, and then (11) leads to an additional restriction for α:

$$\frac{A^4}{(1+A^2)^2} \leqslant \alpha^2 \leqslant \frac{A^2}{1+A^2}. \tag{12}$$

Equations (10) and (11) show that the right boundary of the allowed area is the same but the left boundary depends on the value of the wave amplitude. If $A < 1$ the left boundary is the left boundary of (10). If $A \geqslant 1$ the left boundary is the left boundary of (12).

The general form of the dispersion relations for the waves φ_1 and φ_2 is

$$\gamma = \gamma(\alpha, \beta, A). \tag{13}$$

The dependence on the amplitude A shows that the waves are nonlinear, and the new feature here is the dependence of the parameter α, i.e. on the spatial characteristics of the wave in the direction perpendicular to the direction of propagation.

The dispersion relation of the wave φ_1 is

$$\gamma = \sqrt{\alpha^2 + \beta^2 - \frac{A^2}{1+A^2}}. \tag{14}$$

If we replace the minimum and maximum values of α from (9) we can obtain the allowed frequency area for the wave (2):

$$\sqrt{\beta^2 - \frac{A^4}{(1+A^2)^2}} \leqslant \gamma \leqslant \sqrt{\beta^2 - \frac{A^2}{(1+A^2)^2}}. \tag{15}$$

Equation (15) shows that a minimum value for the wavenumber exists:

$$\beta \geqslant \beta_{\min} = \frac{A^2}{1+A^2}. \tag{16}$$

The dispersion relation for the wave φ_2 is

$$\gamma = \sqrt{\beta^2 - \frac{\alpha^2}{A^2}}. \tag{17}$$

Replacing α with its minimum and maximum values, we have the restrictions

$$A < 1 \rightarrow \sqrt{\beta^2 - \frac{1}{1+A^2}} \leqslant \gamma \leqslant \sqrt{\beta^2 - \frac{1}{(1+A^2)^2}} \tag{18}$$

$$A \geqslant 1 \rightarrow \sqrt{\beta^2 - \frac{1}{1+A^2}} \leqslant \gamma \leqslant \sqrt{\beta^2 - \frac{A^2}{(1+A^2)^2}}. \tag{19}$$

In figure 1 the dependence $\beta_{\min}(A)$ and in figure 2 the allowed and forbidden areas for the parameter α are presented. In figure 3 is presented the dependence $\gamma(A)$ for fixed values of the wavenumber β.

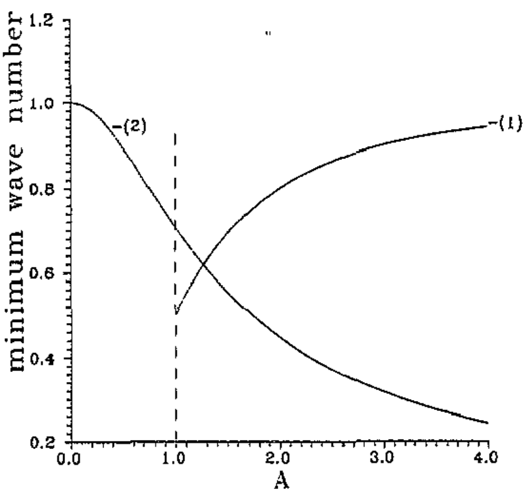

Figure 1. The dependence of the minimum wavenumber for the waves φ_1 and φ_2 on the wave amplitude: (1), minimum wavenumber fo the wave φ_1; (2), minimum wavenumber for the wave φ_2.

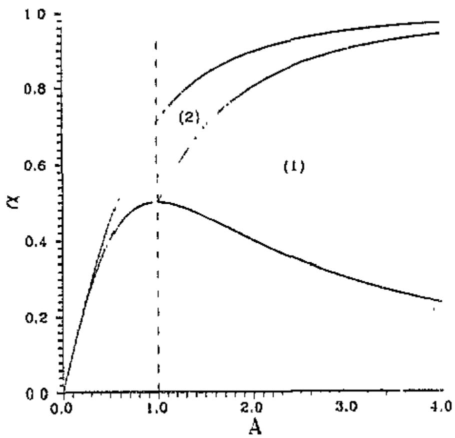

Figure 2. The allowed areas for the parameter α of the waves φ_1 and φ_2: (1), allowed area for the wave φ_1; (2) allowed area for the wave φ_2.

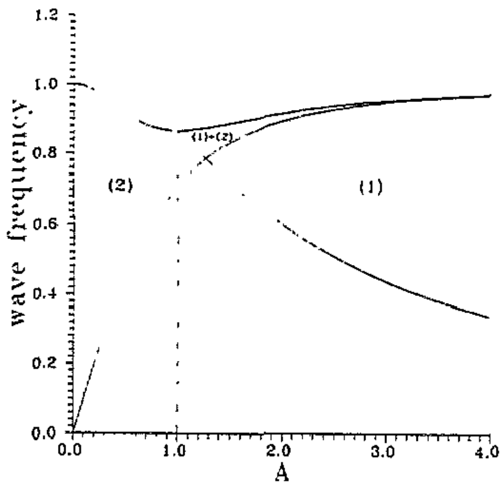

Figure 3. The allowed areas for the frequency of the waves φ_1 and φ_2: (1), allowed area for the wave φ_1; (2), allowed area for the wave φ_2.

A physical system described by the (2+1)-dimensional sine–Gordon equation is the two-dimensional tunnelling Josephson junction [11]. Here φ is the phase difference between the wavefunctions of the electrons in the superconductors of the junction. If the superconductors are made of the same material, the dielectric layer of the junction is parallel to the plane $0yz$, and the magnetic field possesses components only in the direction of the axes $0y$ and $0z$. The Josephson current density and the components of the electromagnetic field in the dielectric layer are [12]

$$j = j_{\max} \sin(\varphi) \tag{20}$$

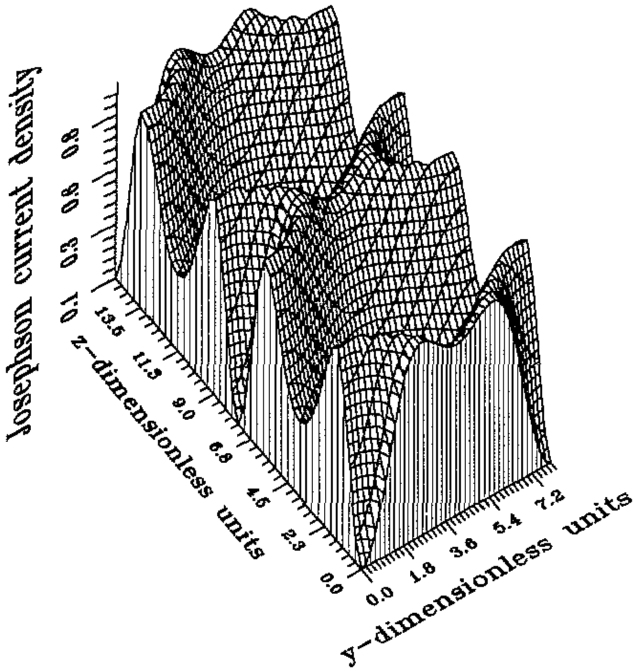

Figure 4. The Josephson current density distribution (units of j/j_{max}) for the self-consistent phase difference distribution φ_1^*. The sizes of the junction are $a = 8$ Josephson lengths and $b = 15$ Josephson lengths. The parameters of the distribution are $n = 1$, $m = 2$.

$$E_x = \frac{\hbar\omega_J}{2el}\frac{\partial\varphi}{\partial t} \qquad E_y = E_z = 0 \tag{21}$$

$$H_x = 0 \qquad H_y = -\frac{\hbar c}{2ed\lambda_J}\frac{\partial\varphi}{\partial z} \qquad H_z = \frac{\hbar c}{2ed\lambda_J}\frac{\partial\varphi}{\partial y} \tag{22}$$

where j_{max} is the amplitude of the current density, e is the charge of the electron, l is the size of the dielectric layer. $d = l + 2\lambda$, where λ is the London penetration depth.

$$\lambda_J = \left(\frac{\hbar c^2}{8\pi d e j_{max}}\right)^{1/2} \qquad \omega_J = \left(\frac{8\pi e l j_{max}}{\hbar\varepsilon}\right)^{1/2} \tag{23}$$

are the Josephson length and frequency and ε is the permittivity of the dielectric layer. The scales in the case of the tunnelling junction are: one dimensionless unit for length corresponds to one Josephson length in dimension units, and one dimensionless unit for frequency corresponds to one Josephson frequency.

The solutions investigated here describe a new class of phase, current and electromagnetic waves in a large two-dimensional junction. These waves need two spatial dimensions for their propagation, i.e. they are not solutions of the (1+1)-dimensional sine–Gordon equation that describes the well known waves (in this case solitons) in the long one-dimensional Josephson junction [13, 14]. The phase velocity of the two-dimensional running waves in [14] is greater than the Swihart velocity $v_S = \lambda_J\omega_J$ [15] while the phase velocity of the waves investigated here is smaller than the Swihart velocity.

The Josephson waves investigated here possess another interesting application. It is well known that the magnetic field of the Josephson current strongly influences the phase difference in the dielectric layer of a large junction. This leads to a self-restriction of the current, and the average Josephson current density cannot reach its maximum theoretical value j_{max}. In the case of a long one-dimensional junction without an electric field ($\partial\varphi/\partial t = 0$), Ferrel and Prange [16, 12] have obtained distributions of the Josephson current and magnetic field in the dielectric layer of the junction and have shown that the current is restricted to the small area near the boundary of the junction, i.e. the long junction can have Meisner-like properties.

In the two-dimensional case the possible distributions of the phase difference, of the Josephson current and of the magnetic field can be obtained on the basis of the solutions of the (2+0)-dimensional sine–Gordon equation. The possible self-consistent distributions lead to different kinds of behaviour of φ on the boundaries of the tunnelling junction. Several kinds of solutions of the (2+0)-dimensional sine–Gordon equation [17–19] and the phase distributions corresponding to the open circuit along the four edges of a rectangular junction have been investigated [19]. If $\gamma = 0$ in (2*a*) and (3*a*) from the running $(2+1)$-dimensional waves φ_1 and φ_2, then as particular cases can be obtained the time-independent two-dimensional phase distributions φ_1^* and φ_2^*. Using the characteristic relations for the $(2+1)$-dimensional sine–Gordon equation [9] one can easily show that if $\alpha = 0$ or $\beta = 0$, φ_1^* and φ_2^* are reduced to the Ferrel and Prange distributions, i.e. here we investigate more general distributions in the large junction that in practice is a two-dimensional system and is not a one-dimensional one.

The distributions φ_1^* and φ_2^* describe doubly periodic cell phase, current and magnetic field structures. Let us impose boundary conditions the same as in [19] on the finite rectangular junction whose dimensionless sizes in the direction of the axes $0y$ and $0z$ are a and b:

$$\left.\frac{\partial\varphi}{\partial y}\right|_{y=0,a} = 0 \qquad \left.\frac{\partial\varphi}{\partial z}\right|_{z=0,b} = 0. \tag{24}$$

In this case we have an open circuit along the four edges of the junction, and the parameters of the distributions depend on two natural numbers: n and m. For both of the structures the parameters α and β connected with the periods of the structures in the corresponding directions ($T_y = 2K(k_1)/\alpha$; $T_z = 2K(k_2)/\beta$) are

$$\alpha_n = \frac{2n}{a}K(k_{1nm}) \qquad n = 1, 2, 3\ldots \tag{25}$$

$$\beta_m = \frac{2m}{b}K(k_{2nm}) \qquad m = 1, 2, 3, \ldots. \tag{26}$$

Here $K(k_{1,2})$ is the complete elliptic integral of the first kind. The other parameters of the distribution φ_1^* are

$$A_{nm}^2 = 4\frac{(n^2/a^2)K^2(k_{1nm}) + (m^2/b^2)K^2(k_{2nm})}{1 - 4[(n^2/a^2)K^2(k_{1nm}) + (m^2/b^2)K^2(k_{2nm})]} \tag{27}$$

$$k_{1nm}^2 = 1 - \frac{(m^2/b^2)K^2(k_{2nm})\{1 - 4[(n^2/a^2)K^2(k_{1nm}) + (m^2/b^2)K^2(k_{2nm})]\}}{(4n^2/a^2)K^2(k_{1nm})[(n^2/a^2)K^2(k_{1nm}) + (m^2/b^2)K^2(k_{2nm})]} \tag{28}$$

$$k_{2nm}^2 = 1 - \frac{(n^2/a^2)K^2(k_{1nm})\{1 - 4[(n^2/a^2)K^2(k_{1nm}) + (m^2/b^2)K^2(k_{2nm})]\}}{(4m^2/b^2)K^2(k_{2nm})[(n^2/a^2)K^2(k_{1nm}) + (m^2/b^2)K^2(k_{2nm})]}. \tag{29}$$

Table 1. Parameters of the self-consistent distribution φ_1^*. a and b are the sizes of the junction in Josephson lengths, n and m are the parameters of the distribution, ν is the coefficient of the self-restriction of the superconducting current.

a	b	n	m	ν
8	15	1	2	0.558
26	24	2	4	0.538
33	22	3	3	0.583
74	111	10	16	0.659

Here the following important restriction must be satisfied:

$$\frac{n^2}{a^2}K^2(k_{1nm}) + \frac{m^2}{b^2}K^2(k_{2nm}) \leqslant \tfrac{1}{4}. \tag{30}$$

The parameters of the distribution φ_2^* are

$$A_{nm}^2 = \frac{b^2n^2K^2(k_{1nm})}{a^2m^2K^2(k_{2nm})} \tag{31}$$

$$k_{1nm}^2 = \frac{[(4n^2/a^2)K^2(k_{1nm}) + (4m^2/b^2)K^2(k_{2nm})]^2 - (4m^2/b^2)K^2(k_{2nm})}{(4n^2/a^2)K^2(k_{1nm})[(4n^2/a^2)K^2(k_{1nm}) + (4m^2/b^2)K^2(k_{2nm})]} \tag{32}$$

$$k_{2nm}^2 = 1 - \frac{[1 - (4n^2/a^2)K^2(k_{1nm}) - (4m^2/b^2)K^2(k_{2nm})](4n^2/a^2)K^2(k_{1nm})}{(4m^2/b^2)K^2(k_{2nm})[(4n^2/a^2)K^2(k_{1nm}) + (4m^2/b^2)K^2(k_{2nm})]}. \tag{33}$$

Here the restriction (30) must also be satisfied, and this restriction leads to maximum values of the parameters n and m. Let us replace the elliptic integrals of the first kind in (30) with their minimum value $\pi/2$. Then

$$1 \leqslant n \leqslant \frac{a}{\pi}\sqrt{1 - \frac{\pi^2}{b^2}} \tag{34}$$

$$1 \leqslant m \leqslant \frac{b}{\pi}\sqrt{1 - \frac{\pi^2}{a^2}}. \tag{35}$$

Equations (34) and (35) lead to restrictions on the size of the junction—the size in both the directions 0y and 0z must be greater than π Josephson lengths because the expressions in the roots of (34) and (35) are smaller than 1. Then we can conclude that the distributions φ_1^* and φ_2^* cannot exist in the small junction or in a large junction whose size is smaller than π Josephson lengths.

One can easily see that (28) and (29) or (32) and (33) are nonlinear algebraic equations for the modules $k_{1,2}$. This means that the periods of each structure $T_y = 2K(k_{1nm})/\alpha_{nm}$ and $T_z = 2K(k_{2nm})/\beta_{nm}$ are connected, i.e. the distributions φ_1^* and φ_2^* are self-consistent. The modules of the Jacobi elliptic functions can be obtained numerically from the above systems of equations, and then one can also obtain the other parameters of the structures. The obtention of the above parameters allow us to calculate the average Josephson current density if a, b, n and m are given:

$$j_a = \frac{1}{ab}\int_0^a \mathrm{d}y \int_0^b \mathrm{d}z[j_{\max}\sin\varphi_{nm}^*(y,z)] = j_{\max}\nu_{nm}. \tag{36}$$

The obtained values for the parameter ν allow us to study the self-restriction of the Josephson current. In table 1 are calculated the parameters of the self-consistent distribution φ_1^* for some values of the parameters a, b, m, n. The form of the spatial distribution of the Josephson current density for the distribution φ_1^* is presented in figure 4.

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